PRETOPOLOGICAL OPERATORS FOR GRAY-LEVEL IMAGE ANALYSIS

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Abstract. This paper deals with new operators for gray-level image analysis. These operators are based on concepts of pretopology and they extend mathematical morphology operators. Instead of using one structuring element, these new operators use a basis of several structuring elements. If this basis is composed of only one element, these operators are equivalent to mathematical morphology ones. This article presents the pretopological representation space and four pretopological structures of operators. Relations between these new operators and the corresponding morphological operators are described and compared. Properties and examples are displayed.

Keywords: Image analysis, Pretopology, mathematical morphology, dilation, erosion, pseudoclosure, interior.

1. Introduction

The mathematical morphology, which has been developed by G.Matheron [12] and J.Serra [15, 17], is based on the use of one structuring element to transform images [16, 6, 14, 9, 13]. Erosion and dilation operators have first been defined on binary images. Then, they have been extended to gray-level images with the help of one functional structuring element [18, 8, 7].
In this work, we build a pretopological space in view to create new gray-level images operators defined by a basis of structuring elements instead of one structuring element. The notion of pretopology has been developed in order to operate on discrete systems [1, 5, 11, 4, 3]. It is an extension of topology, in particular idempotency of closure operators isn't assumed. Some first results have been given on binary images [1], then M.Lamure [10] has proposed to increase these results to gray-level images. He has built a pretopological space according to the decomposition of a 256 gray-level image to 8 levels corresponding to the binary writing of 256. Thus, in that case, a gray-level transformation corresponds to 8 binary ones.
In this paper, we propose a new pretopological structure in view to define and

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operate on gray-level images. In this new working space, we're able to define all mathematical morphology operators and create some new operators with the help of a basis of several structuring elements [2].

2. Gray-level morphology

2.1. Minkowski operators

Let E be an euclidian space and let $P(E)$ be the set of parts of E. In binary image analysis, E is equal to $\mathbb{Z}^2$ or a part of $\mathbb{Z}^2$.

Definition 1. *Let A and B be 2 elements of $P(E)$. The Minkowski addition between A and B is defined by:*

$$A \oplus B = \{a + b \ / \ a \in A \text{ and } b \in B\}$$

Definition 2. *Let $A \in P(E)$, we note $A(x)$ the translated of A by x:*

$$A(x) = A \oplus \{x\} = \{a + x \ / \ a \in A\}$$

Definition 3. *Let $A \in P(E)$, we note $\check{A}$ the symmetric of A by o (o origin of E):*

$$\check{A} = \{-a \ / \ a \in A\}$$

Definition 4. *The Minkowski subtraction is the dual operator of the addition one. Let A and $B \in P(E)$. The Minkowski subtraction between A and B is defined by:*

$$A \ominus B = (A^c \oplus B)^c$$

$\forall A \in P(E)$, A^c is the complementary set of A in E ($A^c = \{x \in E \ / \ x \notin A\}$).

Definition 5. *Let A and $B \in P(E)$:*

$$(1) \quad A \oplus B = \bigcup_{a \in A} B(a) = \bigcup_{b \in B} A(b) = \{z \in E / A \cap (\check{B})(z) \neq \emptyset\}$$

$$(2) \quad A \ominus B = \bigcap_{b \in B} A(b) = \{z \in E / (\check{B})(z) \subseteq A\}$$

According to these notations, mathematical morphology operators can be defined in the following sections.

2.2. Space of gray-level images

Gray-Level Morphology is an extension of binary morphology (in $\mathbb{Z}^2$) to the 3D space $\mathbb{Z}^3$. A gray-level image is defined by a function f on $E \subset \mathbb{Z}^2$ into the discrete interval $[0, \ldots, N_g - 1]$ which corresponds to the N_g gray-level (see figure 1). The function f is a surface of the space $\mathbb{Z}^3$. The set of points under this surface is called the umbra of the image f, this set is denoted $U(f)$ and defined by[18, 7]:

$$U(f) = \{(x, m) \in \mathbb{Z}^3 \ / \ x \in E, m \leq f(x)\}$$

The set $U(f)$ corresponds to a unique function g:

$$g(x) = \sup\{m \ / \ (x, m) \in U(f)\}$$

this function g is equal to f.

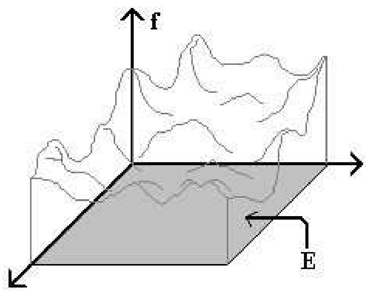

Figure 1: A gray-level image is defined by a function f on E

With these umbras, a gray-level image operation is a sets operation on the corresponding umbra.

In gray-level morphology, we consider functional structuring elements. A structuring element is defined as a 'small' image with a support $B \subset \mathbb{Z}^2$ and a gray-level function h defined on B into $[0, \ldots, N_g - 1]$. We note (B, h) this structuring element (see figure 2). (B, h) is a set of $\mathbb{Z}^3$ which contain the origin $(0, 0, 0)$. The origin $(0, 0, 0)$ corresponds to the origin of the support B. We note h_x the function h defined on $B(x)$:

$$\forall x \in \mathbb{Z}^2, \forall \in B(x), \ h_x(y) = h(y - x)$$

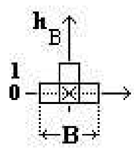

Figure 2: A functional structuring element (B, h)

2.3. Erosion

The erosion of the umbra of the function f, $E_{(B,h)}(U(f))$, with the help of the structuring element (B,h) gives a new set which corresponds to the umbra of a function f', $U(f')$ (see left image of figure 3):

$$E_{(B,h)}(U(f)) = U(f) \ominus \check{B} = U(f')$$

$E_{(B,h)}(U(f)) = U(f')$ is included in $U(f)$.
This is a set theory definition in $\mathbb{Z}^3$. In view to determine the function f', we can compute:

$$\forall x \in E,\ f'(x) = E_{(B,h)}(f)(x) = \sup\{0, \inf_{y \in B(x)} \{f(y) - h_x(y)\}$$

If the structuring element is a non-functional one ($h = 0$), then:

$$\forall x \in E,\ f'(x) = E_B(f)(x) = \inf_{y \in B(x)} \{f(y)\}$$

2.4. Dilation

In the same way, it's possible to create the dilation of an image f with a structuring element (B,h) as the dilation of its umbra $U(f)$ (see right image of figure 3):

$$D_{(B,h)}(U(f)) = U(f) \oplus \check{B}$$

The function corresponding to the dilated image is:

$$\forall x \in E,\ D_{(B,h)}(f)(x) = \inf\{N_g - 1, \sup_{y \in B(x)} \{f(y) + h_x(y)\}\}$$

If the structuring element is a non-functional one, we have:

$$\forall x \in E,\ D_B(f)(x) = \sup_{y \in B(x)} \{f(y)\}$$

3. Gray-level pretopological space

3.1. The Lattice $(S, \leq)$

In the previous section, erosion and dilation have been defined, on one side with set theory in $\mathbb{Z}^3$ with the help of the Minkowski addition, and on the other side, with functions. In the following, we'll consider a gray-level image like a function and not like a set.

Let S be the set of all functions defined on $E \subset \mathbb{Z}^2$ into the discret interval $[0, \ldots, N_g - 1]$. We suppose that E and N_g are finite, thus the cardinality of S is: $\|S\| = N_g^{\|E\|}$.

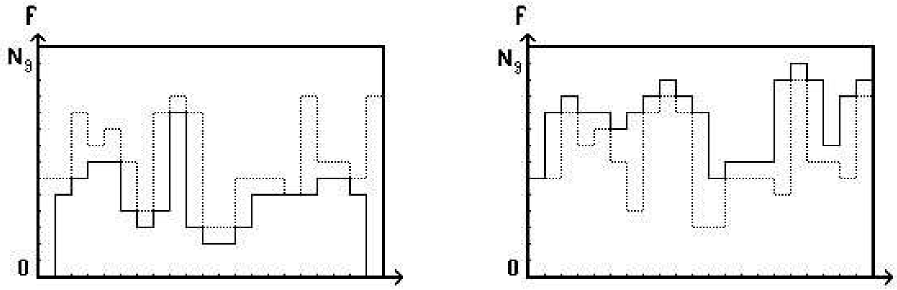

Figure 3: Erosion $E_{(B,h_B)}(f)$ of f and dilation $D_{(B,h_B)}(f)$ of f.

On S, we can define different operations:

1. $\forall f, g \in S, f \leq g \iff \forall x \in E,\ f(x) \leq g(x)$
2. $\forall f, g \in S, f \vee g = \sup\{f, g\}$
3. $\forall f, g \in S, f \wedge g = \inf\{f, g\}$
4. $\forall f, g \in S, f - g = \sup\{0, f - g\}$
5. $\forall f, g \in S, f + g = \inf\{N_g, f + g\}$

According to these definitions, we can easily prove that:

1. $f = N_g - 1$ contains all functions of S: $\forall g \in S,\ g \leq N_g - 1$
2. $f = 0$ is contained in all functions of S: $\forall g \in S,\ 0 \leq g$
3. $f \vee g$ is the smallest element of S above f and g, it always exist
4. $f \wedge g$ is the greatest element of S under f and g, it always exist too
5. For all non empty part P of S, we have: the supremum $\vee P = \sup\limits_{f \in P}\{f\}$ and the infimum $\wedge P = \inf\limits_{f \in P}\{f\}$.

These properties prove that $(S, \leq)$ is a complete lattice (all parts of S has a supremum and an infimum). $\vee$ and $\wedge$ are idempotent, commutative and associative operations, but an element of S has no complementary in S.

3.2. Pretopology on $(S, \leq)$

To define a pretopological structure on a space K, we build two dual operators a and i defined on parts of K into parts of K. But here, on a lattice a and i will be defined from S into S and not from parts of S into parts of S. Moreover, as elements of this lattice have no complementary, we can define dual operators a and i like in a classical pretopological space. That's the reason why, we'll build a generalized pretopological space where a and i are not necessary dual.

In this section, some basic definitions, properties and theoretical results of pretopological structure on finite lattices are explained (proofs can be found in [2]). For that, we'll use the lattice $(S, \leq)$.

Definition 6. *Let a and i two operators defined on S into itself such as:*

1. $\forall f \in S,\ f \leq a(f)$
2. $a(0) = 0$ *(0 is a function f such as $\forall x \in E,\ f(x) = 0$)*
3. $\forall f \in S,\ i(f) \leq f$
4. $i(N_g - 1) = N_g - 1$ *($N_g - 1$ is a function f such as $\forall x \in E,\ f(x) = N_g - 1$)*

a and i define a **pretopological structure** *$s = (a, i)$ on the lattice $(S, \leq)$. $a(f)$ is called the* **pseudoclosure** *of f and $i(f)$ is called the* **interior** *of f.*
If one of the conditions (2) and (4) are not verified then, the pretopological structure s will be called a **weak pretopological structure**.

We can prove that if $s = (a, i)$ is a pretopological structure (weak or not) on $(S, \leq)$, then:

1. $a(N_g - 1) = N_g - 1$
2. $i(0) = 0$

Definition 7. *Let $s = (a, i)$ be a pretopological structure (weak or not) on $(S, \leq)$. This structure is called a V pretopological one if and only if:*

$$\forall f, g \in S \text{ with } f \leq g, \text{ then } \begin{cases} a(f) \leq a(g) \\ i(f) \leq i(g) \end{cases}$$

Definition 8. *Let $s = (a, i)$ be a pretopological structure (weak or not) on $(S, \leq)$. This structure is called a V_D pretopological one if and only if:*

$$\forall f, g \in S, \begin{cases} a(f \vee g) = a(f) \vee a(g) \\ i(f \wedge g) = i(f) \wedge i(g) \end{cases}$$

We can easily prove, that a V_D pretopological structure (weak or not) is a V one.

On a lattice, like $(S, \leq)$, it is possible to build several pretopological structures. In view to compare different possible structures, the following definitions needed:

Definition 9. *Let $PS((S, \leq))$ be the set of pretopological structures (weak or not) on $(S, \leq)$.*
Let $s_1 = (a_1, i_1)$ and $s_2 = (a_2, i_2)$ be two elements of $PS((S, \leq))$.
The structure s_1 is finer than the structure s_2 (we note $s_1 \lhd s_2$) if and only if $\forall f \in S,\ a_1(f) \leq a_2(f)$ and $i_2(f) \leq i_1(f)$. $\lhd$ is an order relation on $PS((S, \leq))$.

We can define too the notion of closed element, closure, opened element and opening on $(S, \leq)$:

Definition 10. *Let $s = (a, i)$ be an element of $PS((S, \leq))$. We have:*

1. *f is closed $\iff a(f) = f$*
2. *f is opened $\iff i(f) = f$*
3. *$F(f)$ is the closure of $f \iff F(f) = \wedge\{g \ / \ f \leq g$ and g closed $\}$*
4. *$O(f)$ is the opening of $f \iff O(f) = \vee\{g \ / \ g \leq f$ and g opened $\}$*

From these definitions, we can prove that:

1. if f is closed then $F(f) = f$
2. if f is opened then $O(f) = f$
3. if s is not weak, the functions 0 and $N_g - 1$ are closed and opened

If s is a V_D pretopological structure then the supremum of two closed elements of S is closed and the infimum of two opened elements of S is opened. Moreover, if s is a V pretopological structure (weak or not) then the closure and the opening of each element of S exist.

If s is a V_D pretopological (weak or not) structure then:

$$\forall f \in S, \begin{cases} \exists n \in \mathbb{N} \text{ finite, such as: } F(f) = a^n(f) \\ \exists m \in \mathbb{N} \text{ finite, such as: } O(f) = i^m(f) \end{cases}$$

4. Gray-level pretopology operators

We build 4 pretopological structures on $(S, \leq)$. These structures correspond to new image operators and those well known of mathematical morphology.

In this way, we need a basis of structuring elements:

$$\begin{aligned} \forall x \in E, \ B(x) &= \{(B^1(x), h^1), \ldots, (B^n(x), h^n)\} \\ &= \{(B^j(x), h^j)\}_{j \in J} \ , \ J = \{1, \ldots, n\} \end{aligned}$$

This basis is composed of n structuring elements (n finite) used to build pseudoclosure and interior on $(S, \leq)$.

4.1. Fine structure

Let a and i be two operators on S into S such as: $\forall f \in S, \ \forall x \in E$,

$$a(f(x)) = f_a(x) = \inf[N_g - 1, \inf_{j \in J}(\sup_{y \in B^j(x)} (f(y) + h^j_x(y)))]$$

$$i(f(x)) = f_i(x) = \sup[0, \sup_{j \in J}(\inf_{y \in B^j(x)} (f(y) - h^j_x(y)))]$$

We note PS_{fine} the structure defined by a and i. To show that $s = (a, i)$ is a pretopological structure on $(S, \leq)$ is enough to prove that $f_i \leq f \leq f_a$.

In the main cases, PS_{fine} is a weak structure, but if the structural elements are non functional ones then PS_{fine} is not a weak structure and then: $a(0) = 0$ and $i(N_g - 1) = N_g - 1$.

We can easily prove that PS_{fine} is a V pretopological structure.

4.2. Mean structure

Let a and i be two operators on S into S such as: $\forall f \in S,\ \forall x \in E$,

$$a(f(x)) = f_a(x) = \inf[N_g - 1, \frac{1}{card(J)} \sum_{j \in J} (\sup_{y \in B^j(x)} (f(y) + h_x^j(y)))]$$

$$i(f(x)) = f_i(x) = \sup[0, \frac{1}{card(J)} \sum_{j \in J} (\inf_{y \in B^j(x)} (f(y) - h_x^j(y)))]$$

We note PS_{mean} the structure defined by a and i. It's a V pretopological one.

4.3. Median structure

Let a and i be two operators on S into S such as: $\forall f \in S,\ \forall x \in E$,

$$a(f(x)) = f_a(x) = \inf[N_g - 1, med_{j \in J}(\sup_{y \in B^j(x)} (f(y) + h_x^j(y)))]$$

$$i(f(x)) = f_i(x) = \sup[0, med_{j \in J}(\inf_{y \in B^j(x)} (f(y) - h_x^j(y)))]$$

This structure $s = (a, i)$ is called PS_{medi} and it's a V pretopological one.

4.4. Deep structure

Let a and i be two operators on S into S such as: $\forall f \in S,\ \forall x \in E$,

$$a(f(x)) = f_a(x) = \inf[N_g - 1, \sup_{j \in J}(\sup_{y \in B^j(x)} (f(y) + h_x^j(y)))]$$

$$i(f(x)) = f_i(x) = \sup[0, \inf_{j \in J}(\inf_{y \in B^j(x)} (f(y) - h_x^j(y)))]$$

This structure $s = (a, i)$ is called PS_{deep} and it's a V pretopological one.

4.5. Structure comparison

According to definition 9, we compare these 4 pretopological structures. $\lhd$ is an order relation between pretopological structures on a same space. This order displays a notion of fineness between structures.

It's easy to prove that:

$$\inf[N_g - 1, \inf_{j\in J}(\sup_{y\in B^j(x)}(f(y)+h_x^j(y)))] \leq \inf[N_g - 1, \tfrac{1}{card(J)}\sum_{j\in J}(\sup_{y\in B^j(x)}(f(y)+h_x^j(y)))]$$

$$\sup[0, \sup_{j\in J}(\inf_{y\in B^j(x)}(f(y)-h_x^j(y)))] \geq \sup[0, \tfrac{1}{card(J)}\sum_{j\in J}(\inf_{y\in B^j(x)}(f(y)-h_x^j(y)))]$$

$$\inf[N_g - 1, \inf_{j\in J}(\sup_{y\in B^j(x)}(f(y)+h_x^j(y)))] \leq \inf[N_g - 1, med_{j\in J}(\sup_{y\in B^j(x)}(f(y)+h_x^j(y)))]$$

$$\sup[0, \sup_{j\in J}(\inf_{y\in B^j(x)}(f(y)-h_x^j(y)))] \geq \sup[0, med_{j\in J}(\inf_{y\in B^j(x)}(f(y)-h_x^j(y)))]$$

These relations prove that $SP_{fine} \lhd SP_{mean}$ and that $SP_{fine} \lhd SP_{medi}$.
We have too:

$$\inf[N_g - 1, \sup_{j\in J}(\sup_{y\in B^j(x)}(f(y)+h_x^j(y)))] \geq \inf[N_g - 1, \tfrac{1}{card(J)}\sum_{j\in J}(\sup_{y\in B^j(x)}(f(y)+h_x^j(y)))]$$

$$\sup[0, \inf_{j\in J}(\inf_{y\in B^j(x)}(f(y)-h_x^j(y)))] \leq \sup[0, \tfrac{1}{card(J)}\sum_{j\in J}(\inf_{y\in B^j(x)}(f(y)-h_x^j(y)))]$$

$$\inf[N_g - 1, \sup_{j\in J}(\sup_{y\in B^j(x)}(f(y)+h_x^j(y)))] \geq \inf[N_g - 1, med_{(}\sup_{y\in B^j(x)}(f(y)+h_x^j(y)))]$$

$$\sup[0, \inf_{j\in J}(\inf_{y\in B^j(x)}(f(y)-h_x^j(y)))] \leq \sup[0, med_{j\in J}(\inf_{y\in B^j(x)}(f(y)-h_x^j(y)))]$$

Thus, it proves that $SP_{mean} \lhd SP_{deep}$ and $SP_{medi} \lhd SP_{deep}$.
As $\lhd$ is an order relation, we have $SP_{fine} \lhd SP_{deep}$.

The five following images show results of the four operators a on figure 4 with the help of the basis given by the figure 5: figure 6 displays Fine pseudoclosure and Mean one, and figure 7 displays Median pseudoclosure and Deep one.

Figure 4: Original gray-level image

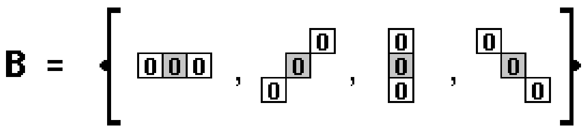

Figure 5: Basis of 4 non-functional structuring elements ($h^j = 0$ $\forall j \in \{1, \ldots, 4\}$)

For all pseudoclosure operators a defined on $(S, \leq)$, the gray-level are increased or not modified (they can't be reduced). Then, if a pseudoclosure a_1 is finer than a pseudoclosure a_2 ($a_1 \lhd a_2$), the increasing of gray-level by a_1 is less important than the a_2 one. Thus, the corresponding image is less light with a_1 than with a_2. In figures 6 and 7, we can see that image given by the pseudoclosure of PS_{fine} is darker than the three other ones. On the over hand, the deep pseudoclosure is lighter than the three other ones.

Figure 6: Fine pseudoclosure following by Mean pseudoclosure

Figure 7: Median pseudoclosure following by Deep pseudoclosure

5. Generalization of mathematical morphology

A pretopological structure is composed of a pseudoclosure operator a and an interior one i. We easily show that these both operators can recreate those of mathematical morphology (dilation and erosion). For that, we study the SP_{fine} structure.

As said before, dilation and erosion use only one structuring element. Let's write the pseudoclosure and the interior with a basis composed of only one structuring element, we note the corresponding functions g_a and g_i:

$$\begin{cases} g_a(x) = \inf[N_g - 1, \sup\limits_{y \in B(x)} (g(y) + h_x(y))] \\ g_i(x) = \sup[0, \inf\limits_{y \in B(x)} (g(y) - h_x(y))] \end{cases}$$

g_a and g_i are exactly definitions of dilation and erosion. This writing proves that if there's only one element in the basis, the pseudoclosure is equal to the dilation and the interior is equal to the erosion. For example, if we use the basis displayed by figure 8 composed of one element, the pseudoclosure by SP_{fine} corresponds to the dilation by the element of this basis.

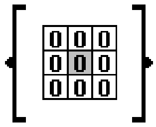

Figure 8: A non-functional structuring element composed of the origin and its 8 neighbours

But, if there's more than one structuring elements in the basis we can create new image transformations. For example, figure 9 shows results given by the SP_{fine} pseudoclosure with the basis displayed by figure 8 and with the basis displayed by figure 5 which is the decomposition of the element of figure 8.

Figure 9: Fine pseudoclosure with a basis composed of one element (B, h) following by Fine pseudoclosure with the basis of figure 5

We can see that the second transformation (obtained from the basis displayed by 5), noted f_2, is finer than the first one (obtained from the basis displayed by 8), noted f_1. If we note f the original image, we have: $f \leq f_2 \leq f_1$. This example shows the difference between a dilation and a new pretopological transformation realized with the help of a basis which is the decomposition of the structuring element used in the dilation.
We note that the use of the deep pseudoclosure with the basis of figure 5 corresponds to the dilation with the only one element of figure 8 because the deep pseudoclosure keeps the supremum of the supremum of each structuring element of the basis. It depends of the pretopological structure built.

6. Others pretopological operators

Some other pretopological operators can be built; three of them are briefly studied: the Edge, the Orle and the Boundary.

First, figure 10 displays the basis used with its two elements and the image f on which will be applied operators.

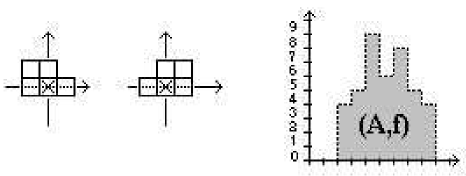

Figure 10: The two elements of the basis used on the image (A, f)

Definition 11. *The* **Edge** *is an operator defined on S into itself such as:*

$$\forall g \in S,\ b(g) = g - i(g)$$

The Edge-image of the image f corresponds to the gray pixels in the figure 11.

Definition 12. *The* **Orle** *is an operator defined on S into itself such as:*

$$\forall g \in S,\ o(g) = a(g) - g$$

The Orle-image of the image f corresponds to the gray pixels in the figure 12.

Definition 13. *The* **Boundary** *is an operator defined on S into itself such as:*

$$\forall g \in S,\ bo(g) = a(g) - i(g)$$

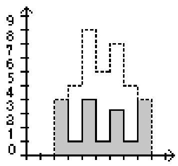

Figure 11: Edge-image of f

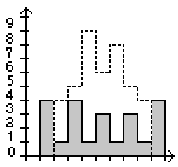

Figure 12: Orle-image of f

The Boundary-image of the image f corresponds to the gray pixels in the figure 13.

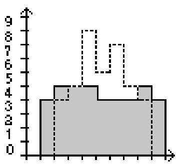

Figure 13: Boundary-image of f

The figure 14 displays these three operators on the original image of figure 4.

7. Conclusion

This work presents pretopological concepts designed for gray-level image analysis. We propose four structures corresponding to four new dilations (pseudo-closures) and four new erosions (interiors). These operators extend mathematical morphology operators in regard with the number of elements used to operate transformations. These new operators create new image transformations finer or deeper than mathematical morphology ones.

Figure 14: Edge, orle and boundary of original image of figure 4

The use of pretopological approach brings many other mathematical tools well known in pretopology (connectivity, continuity, separability, ...). These tools have improved binary image analysis for segmentation problems, area detection, ... ; we'd like to use these tools to improve gray-level image analysis.

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